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Why Unsharp Observables?

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Abstract We discuss why projection valued measures are not sufficient in the description of position and momentum of a one dimensional particle. A satisfactory solution is offered using positive operator measures. We also argue why the relevant positive operator measures, but not all, may be called unsharp observables.

1 Introduction

It is well understood that the description of, for instance, phase and time observables demands the use of positive operator measures (POMs) rather than just projection valued measures (PVMs). There is simply no PVM with desirable properties; especially, there is no PVM which would have the required covariance property.

The case of position and momentum of a one dimensional (massive) particle is different as here covariant PVMs exists. However, even in the description of position and momentum observables POMs are needed. First of all, there is no joint measurement for the canonical position and momentum. Another reason is that the canonical position and momentum lack feasible measurement models.

Evidently, any description of a real experiment uses effects rather than projections as the latter are far too ideal. We emphasize that the POMs mentioned above are not just useful in this sense—they are already required by theoretical reasons. The reasons in the two different kind of situations (phase and time vs. position and momentum) are different, but in both cases the reasons have more foundational than practical flavor.

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Notations and Basic Framework Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ the Banach space of bounded linear operators on $\mathcal{H}$. *States* of the system are represented by (and identified with) positive operators of trace one, and we denote the set of states by $\mathcal{S}(\mathcal{H})$. Each unit vector $\psi \in \mathcal{H}$ defines a one-dimensional projection $\phi \mapsto \langle \psi \mid \phi \rangle \psi$, which we denote by P_ψ. These kind of operators are the extreme elements of the convex set $\mathcal{S}(\mathcal{H})$, and we refer to them as *pure states*.

An *observable* having the Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as outcome space is represented as a positive operator measure $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$, which is normalized in the sense that $\mathsf{E}(\mathbb{R}) = \mathbb{1}$. We will occasionally use the equivalent representation of an observable as a normalized positive mapping $\mathsf{E} : C_c(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$, where $C_c(\mathbb{R})$ is the space of compactly supported continuous functions. The normalization means that $\mathrm{LUB}\{\mathsf{E}(f) \mid f \in C_c(\mathbb{R}), 0 \leq f \leq 1\} = \mathbb{1}$.

Any pair of an observable E and a state T determines a probability measure p_T^{E} through the trace formula

$$p_T^{\mathsf{E}}(X) := \mathrm{tr}[T\mathsf{E}(X)], \quad X \in \mathcal{B}(\mathbb{R}). \tag{1}$$

This is the measurement outcome probability distribution when a measurement of E is performed in the state T.

2 Unsharp Position and Momentum Observables

In the following we are looking for unsharp versions of the canonical position observable Q and the canonical momentum observable P. We consider here only the case of a system moving in the real line $\mathbb{R}$ and hence, we fix $\mathcal{H} = L^2(\mathbb{R}, dx)$. For any pure state $T = P_\psi$, we have $p_T^{\mathsf{Q}}(X) = \int_X |\psi(x)|^2 dx$ and $p_T^{\mathsf{P}}(Y) = \int_Y |\hat{\psi}(x)|^2 dx$. Since Q and P are unitarily related by the Fourier–Plancherel transform, the results we describe for position are easily converted to momentum and vice versa.

2.1 Smearings

Let E be an observable and $k : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ a Markov kernel (i.e. each $k(\cdot, X)$ is a measurable function and each $k(x, \cdot)$ is a probability measure). For any $X \in \mathcal{B}(\mathbb{R})$, define

$$\mathsf{F}(X) = \int k(x, X) d\mathsf{E}(x). \tag{2}$$

Then F is an observable and we say that it is a *smearing* of E. This kind of description of the effect of imprecision has been used, for instance, in [1, 2, 11].

A smearing F of the canonical position Q is a some sort of inaccurate or fuzzy version of Q. However, this does not yet guarantee that F would be an appropriate candidate to represent an unsharp position observable. Namely, an unsharp position should have some characteristic features of position. In other words, it should be possible to identify an unsharp position as an inaccurate version of Q, not of something else.

Example 1 Let λ be a probability measure on $\mathbb{R}$ and define a Markov kernel as $k(x, X) = \lambda(X)$ for every $x \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$. For any observable E, the smearing F given by the formula (2) is the trivial observable $X \mapsto \lambda(X)\mathbb{1}$. Hence, the trivial observable $\lambda\mathbb{1}$ is a smearing of every observable. Although a trivial observable is a smearing of Q, it has lost every characteristic feature of position.

For each $q \in \mathbb{R}$, define a unitary operator $U_q : \mathcal{H} \to \mathcal{H}$ by

$$[U_q \psi](x) = \psi(x - q). \tag{3}$$

An observable E is said to be *translation covariant* if

$$U_q \mathsf{E}(X) U_q^* = \mathsf{E}(X + q) \tag{4}$$

for every $q \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$. The translation covariance is the basic kinematical property of the canonical position Q. It reflects the idea that when the location of the measuring apparatus is changed, the measurement outcome distribution is shifted accordingly.

As shown in [13], each smearing of Q which has the covariance property (4) is of the form

$$\mathsf{Q}_\rho(X) = \int \rho(X - x) d\mathsf{Q}(x), \tag{5}$$

where ρ is a probability measure on $\mathbb{R}$.

Similarly, the basic kinematical property of the canonical momentum observable P is the covariance under velocity boosts. For each $p \in \mathbb{R}$, define a unitary operator $V_p : \mathcal{H} \to \mathcal{H}$ by

$$[V_p \psi](x) = e^{ipx} \psi(x). \tag{6}$$

Each boost covariant smearing of P is of the form

$$\mathsf{P}_\nu(X) = \int \nu(X - x) d\mathsf{P}(x), \tag{7}$$

where ν is a probability measure on $\mathbb{R}$.

The sole requirement of boost covariance without further assumptions is not enough to lead to the form (7). It is interesting that there are even commutative boost covariant observables which are not smearings of P. This is illustrated in the following example.

Example 2 As described in [10, Sect. V.A], each boost covariant observable is defined by a weakly measurable map $x \mapsto h_x$ from $\mathbb{R}$ to the set of unit vectors in $\mathcal{H}$. Fix a unit vector $h \in \mathcal{H}$ and define $h_x = e^{ix^2} h$ for every $x \in \mathbb{R}$. Then the resulting observable E is given by the formula

$$\begin{aligned}[\mathsf{E}(\varphi)\phi](x) &= \int \mathcal{F}^*\varphi(y) e^{-2ixy} e^{iy^2} \phi(x - y) dy \\ &= e^{-ix^2} [\mathcal{F}^*\varphi * (e^{i\cdot^2} \phi(\cdot))](x),\end{aligned}$$

where $\varphi \in C_c(\mathbb{R})$ and $\phi \in L^2(\mathbb{R}, dx)$. It is easy to check that E is a commutative observable. Indeed, for every $\varphi_1, \varphi_2 \in C_c(\mathbb{R})$ and $\phi \in L^2(\mathbb{R}, dx)$, we have

$$\begin{aligned}[\mathsf{E}(\varphi_1)\mathsf{E}(\varphi_2)\phi](x) &= e^{-ix^2} [\mathcal{F}^*\varphi_1 * [\mathcal{F}^*\varphi_2 * (e^{i\cdot^2} \phi(\cdot))]](x) \\ &= e^{-ix^2} [\mathcal{F}^*\varphi_2 * [\mathcal{F}^*\varphi_1 * (e^{i\cdot^2} \phi(\cdot))]](x) \\ &= [\mathsf{E}(\varphi_2)\mathsf{E}(\varphi_1)\phi](x).\end{aligned}$$

However, E does not commute with the unitary operators $U_q, q \neq 0$, and hence not with P. In fact, if $q \in \mathbb{R}$, we have, for $\varphi \in C_c(\mathbb{R})$ and $\phi \in L^2(\mathbb{R}, dx)$,

$$[\mathsf{E}(\varphi)U_q\phi](x) = e^{-i(x-q)^2} \int (\mathcal{F}^* U_{-2q}\varphi)(x-y-q)e^{iy^2}\phi(y)dy$$

and

$$[U_q\mathsf{E}(\varphi)\phi](x) = e^{-i(x-q)^2} \int \mathcal{F}^*\varphi(x-y-q)e^{iy^2}\phi(y)dy.$$

It follows that $\mathsf{E}(\varphi)$ commutes with U_q if and only if

$$(\mathcal{F}^* U_{-2q}\varphi) * (e^{i\cdot^2}\phi(\cdot)) = \mathcal{F}^*\varphi * (e^{i\cdot^2}\phi(\cdot))$$

for all $\phi \in L^2(\mathbb{R}, dx)$. This is equivalent to $U_{-2q}\varphi = \varphi$, which is impossible if $q \neq 0$.

2.2 Relativistic Approach

One can alternatively start from the symmetry inspection of an elementary system moving in the real line. The symmetry group of the system is the group of space translations and velocity boosts. It acts on the Hilbert space $\mathcal{H}$ of the system by means of an irreducible projective unitary representation W. We assume that the system can be in at least two different pure states. We may then without restriction take $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $W_{q,p} = e^{iqp/2}U_qV_p$ for every $(q, p) \in \mathbb{R}^2$.

An observable E is now naturally called a position observable if

$$W_{q,p}\mathsf{E}(X)W^*_{q,p} = \mathsf{E}(X+q) \tag{8}$$

for every $(q, p) \in \mathbb{R}^2$ and $X \in \mathcal{B}(\mathbb{R})$. Compared to (4), here we have the additional requirement of boost invariance. It is proved in [7] that E is a position observable exactly when $\mathsf{E} = \mathsf{Q}_\rho$ for some probability measure ρ.

In order to discuss an additional symmetry property, let $\mathbb{R}_+$ be the set of positive real numbers regarded as a multiplicative group. If $a \in \mathbb{R}_+$ and $X \subseteq \mathbb{R}$, the set $aX = \{ax \mid x \in X\}$ is a dilation, or scaling, of X. In this way, $\mathbb{R}_+$ is a (nontransitive) transformation group on $\mathbb{R}$.

We say that an observable E is *dilation covariant* if there exists a unitary representation D of $\mathbb{R}_+$ such that for all $a \in \mathbb{R}_+$ and $X \in \mathcal{B}(\mathbb{R})$,

$$\mathsf{E}(aX) = D_a^*\mathsf{E}(X)D_a. \tag{9}$$

This means that the observable E has no scale dependence: any scaling of the outcome space gives a unitarily equivalent observable. This is clearly an indication of ideal precision.

The dilation group $\mathbb{R}_+$ has the following unitary representation on $\mathcal{H}$:

$$[D_a\psi](x) = \sqrt{a}\psi(ax). \tag{10}$$

It is directly verified that the canonical position Q is dilation covariant under this representation. Moreover, it was proved in [7, Sect. II.B] that a position observable is dilation covariant exactly when it is a projection valued measure.

The following summarizes the previous discussion.

- A *position observable* is an observable defined on $\mathcal{B}(\mathbb{R})$ which is translation covariant and boost invariant. The position observables are in one-to-one correspondence with the probability measures on $\mathbb{R}$ via the formula (5).
- A *sharp position observable* is a position observable which has only projections in its range. They are exactly the dilation covariant position observables. Each sharp position has the form $X \mapsto U_q^* \mathsf{Q}(X) U_q$ for some $q \in \mathbb{R}$.
- An *unsharp position observable* is a position observable which is not sharp. This means that the corresponding probability measure ρ is not a Dirac measure.

3 Joint Measurements of Unsharp Position and Momentum

A position observable Q_ρ and a momentum observable P_ν are called *jointly measurable* if they have a joint observable G. This means that G is an observable defined on $\mathcal{B}(\mathbb{R}^2)$ and Q_ρ and P_ν are its margins, i.e.,

$$\mathsf{Q}_\rho(X) = \mathsf{G}(X \times \mathbb{R}), \qquad \mathsf{P}_\nu(Y) = \mathsf{G}(\mathbb{R} \times Y)$$

for all $X, Y \in \mathcal{B}(\mathbb{R})$.

In his recent article [17], Werner showed that the question of joint measurability of position and momentum observables can be reduced to the study of covariant phase space observables. This result leads to the complete characterization of jointly measurable pairs of position and momentum observables as explained in [8]. In particular, combining [7, Proposition 5] and [8, Proposition 7], we conclude that for a position observable Q_ρ, the following conditions are equivalent:

(i) The probability measure ρ is absolutely continuous with respect to the Lebesgue measure;
(ii) There is a joint measurement of Q_ρ and some momentum observable.

Also, one can show that if $\mathsf{Q}_\rho(X)$ is a nontrivial projection operator for some $X \in \mathcal{B}(\mathbb{R})$, then Q_ρ is not jointly measurable (or even coexistent) with any momentum observable [8, Proposition 11].

4 Some Remarks on General Definition of an Unsharp Observable

It is customary to refer to projection valued measures as sharp observables. It is then simply a terminological question what is meant by an unsharp observable. A possible choice is that any observable which is not sharp is unsharp. However, it seems that this choice may be misleading and perhaps not the most favorable. To demonstrate the problem, we compare the above definition of unsharp position to another example.

Let Q_ρ be an unsharp position observable and let $X \subset \mathbb{R}$ be an interval with diameter smaller than the diameter of the support of the measure ρ. As shown in [9], there is a number $c < 1$ such that $p^T_{\mathsf{Q}_\rho}(X) < c$ for every state $T \in \mathcal{S}(\mathcal{H})$. This indicates that the observable Q_ρ has intrinsic fuzziness.

Now, let us compare this with the situation of the canonical phase observable M (see e.g. [15] for the definition and basic properties). For any interval $X \subset [0, 2\pi)$ and for any

$c < 1$, there is a state T such that $p^T_{\mathsf{M}}(X) > c$; see, for instance, [12]. This suggests that M is optimally accurate.

Furthermore, it was shown in [3] that M is totally noncommutative in the sense that if $\psi \in \mathcal{H}$ satisfies the equation

$$\mathsf{M}(X)\mathsf{M}(Y)\psi = \mathsf{M}(Y)\mathsf{M}(X)\psi$$

for all $X, Y \in \mathcal{B}([0, 2\pi))$, then $\psi = 0$. Since a smearing of a sharp observable is a commutative observable, this means that there is no sharp observable which could give M as a smearing.

For the above reasons we propose to avoid calling M an unsharp observable even though it is not a projection valued measure.

5 Position Observables and Instruments

We recall the basic framework of quantum measurements as presented, for instance, in [5]. A normal unitary premeasurement $\mathcal{M}$ for a position observable Q_ρ (from now on, simply a "measurement model for Q_ρ") consists of

- a Hilbert space $\mathcal{K}$
- a PVM E, defined on $\mathbb{R}$ and acting on $\mathcal{K}$
- a unit vector $v \in \mathcal{K}$
- a unitary operator U defined on $\mathcal{H} \otimes \mathcal{K}$

such that

$$\langle \phi | \mathsf{Q}_\rho(X)\phi \rangle = \langle U(\phi \otimes v) | I \otimes \mathsf{E}(X) U(\phi \otimes v) \rangle \quad \forall \phi \in \mathcal{H}. \tag{11}$$

The physical interpretation of the above data is the following. The system $\mathcal{S}$ interacts through the *measurement coupling* U with a *measuring apparatus* $\mathcal{A}$ (with associated Hilbert space $\mathcal{K}$) whose *initial state* is P_v. If the initial state of the system is P_ϕ, then the statistics of the observable Q_ρ is obtained through the formula (11), by measuring the *pointer observable* E on the apparatus $\mathcal{A}$.

Any measurement model $\mathcal{M}$ determines an instrument $\mathfrak{I}^{\mathcal{M}}$ (i.e. an operation valued measure $X \mapsto \mathfrak{I}^{\mathcal{M}}_X$ from $\mathcal{B}(\mathbb{R})$ to $\mathcal{L}(\mathcal{T}(\mathcal{H}))^+$) through the formula

$$\mathcal{I}^{\mathcal{M}}_X(T) = \mathrm{tr}_{\mathcal{K}}[U(T \otimes P_v)U^* I \otimes \mathsf{E}(X)]. \tag{12}$$

The instrument $\mathcal{I}^{\mathcal{M}}$ is completely positive, and moreover, each completely positive instrument can be presented in this form [16]. The instrument $\mathcal{I}^{\mathcal{M}}$ encodes the essential information about the measuring process in the sense that it allows to reconstruct both the statistic of the measurement outcomes and the conditional final state of the system.

5.1 von Neumann model

We start by briefly summarizing the von Neumann model of a position measurement; for more details and references, see e.g. [4].

The position of a particle $\mathcal{S}$ (with Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$) is measured by coupling the system with another particle $\mathcal{A}$ (with Hilbert space $\mathcal{K} = L^2(\mathbb{R}, dx)$) through the unitary evolution

$$U = e^{i(Q_{\mathcal{S}} \otimes P_{\mathcal{A}})} = \int_{\mathbb{R}^2} e^{-iqp} d\mathsf{Q}_{\mathcal{S}}(q) \otimes d\mathsf{P}_{\mathcal{A}}(p).$$

Here $\mathsf{Q}_{\mathcal{S}}$ is the canonical position observable of the system and $\mathsf{P}_{\mathcal{A}}$ is the canonical momentum observable of the apparatus. The action of U on decomposable vectors is easily calculated and is given by

$$U(\phi\otimes v)(x,y)=\phi(x)v(y-x). \tag{13}$$

The pointer observable is chosen to be the canonical position observable $\mathsf{Q}_{\mathcal{A}}$ of the apparatus.

The Associated Instrument We assume that the initial state of the system $\mathcal{S}$ is the pure state P_ϕ and hence, the initial state of the combined system is $P_{\phi\otimes v}$. Due to (12), the instrument is completely determined by the matrix element

$$\begin{aligned}\langle\psi|\mathcal{I}_X(P_\phi)\psi\rangle &= \mathrm{tr}[P_\psi\ \mathrm{tr}_{\mathcal{K}}[UP_{\phi\otimes v}U^*(I\otimes\mathsf{Q}_{\mathcal{A}}(X))]]\\ &= \mathrm{tr}[P_\psi\ \mathrm{tr}_{\mathcal{K}}[P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_{\mathcal{A}}(X))]]\\ &= \mathrm{tr}[(P_\psi\otimes I)P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_{\mathcal{A}}(X))].\end{aligned}$$

By (13) we have

$$\begin{aligned}&[P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_{\mathcal{A}}(X))f](x,y)\\ &\quad=\phi(x)v(y-x)\iint\chi_X(y')\overline{\phi(x')v(y'-x')}f(x',y')dx'dy'\end{aligned} \tag{14}$$

for all $f\in L^2(\mathbb{R}^2,d^2x)$. Moreover,

$$\begin{aligned}\langle\psi|\mathcal{I}_X(P_\phi)\psi\rangle &= \mathrm{tr}[P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_A(X))P_\psi\otimes I(I\otimes\mathsf{Q}_A(X))P_{U(\phi\otimes v)}]\\ &= \mathrm{tr}[P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_A(X))L_\psi L_\psi^*(I\otimes\mathsf{Q}_A(X))P_{U(\phi\otimes v)}]\end{aligned}$$

where $L_\psi:\mathcal{K}\to\mathcal{H}\otimes\mathcal{K}$ is the operator defined by $k\mapsto\psi\otimes k$. We then have

$$\langle\psi|\mathcal{I}_X(P_\phi)\psi\rangle=\|P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_A(X))L_\psi\|_{\mathrm{HS}}^2$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. Using equation (14), we conclude that the Hilbert–Schmidt operator $P_{U(\phi\otimes v)}(I\otimes\mathsf{Q}_A(X))L_\psi$ is the integral operator from $\mathcal{K}=L^2(\mathbb{R},dx)$ to $\mathcal{H}\otimes\mathcal{K}\simeq L^2(\mathbb{R}^2,d^2x)$ with kernel

$$\Gamma(x,y;z)=\left(\int\overline{\phi(q)v(z-q)}\psi(q)dq\right)\chi_X(z)\phi(x)v(y-x).$$

Hence, using previous equation, we obtain that

$$\begin{aligned}\langle\psi|\mathcal{I}_X(P_\phi)\psi\rangle &= \|\Gamma\|_{L^2(\mathbb{R}^3)}^2\\ &= \int dq'\overline{\psi(q')}\phi(q')\iint dzdq\,\chi_X(z)\psi(q)\overline{\phi(q)}v(z-q')\overline{v(z-q)}\end{aligned}$$

and finally,

$$(\mathcal{I}_X(P_\phi)\psi)(x)=\phi(x)\iint\overline{\phi(x')}\chi_X(y)v(y-x)\overline{v(y-x')}\psi(x')dx'dy. \tag{15}$$

The associated observable is easily calculated. It is the position observable Q_ρ determined by the probability measure

$$d\rho(x) = |v(x)|^2 dx$$

If $v \in \mathcal{K}$ is not only square integrable but also essentially bounded, then the operator

$$\begin{aligned} K_x : L^2(\mathbb{R}, dx) &\longrightarrow L^2(\mathbb{R}, dx), \\ \psi &\longmapsto (y \mapsto ([K_x\psi](y) = v(x-y)\psi(y))) \end{aligned}$$

is well defined and the formula (15) can be rewritten as:

$$\mathcal{I}_X(P_\phi) = \int_X K_x P_\phi K_x^* dx$$

One of the peculiarities exhibited by the above instrument is the fact that it is translation covariant and boost invariant. This means that for every $q, p \in \mathbb{R}$, $X \in \mathcal{B}(\mathbb{R})$, $T \in \mathcal{S}(\mathcal{H})$, we have

$$\mathcal{I}_{X+q}(T) = U_q V_p \mathcal{I}_X(V_p^* U_q^* T U_q V_p) V_p^* U_q^*, \tag{16}$$

where U and V are the representations defined in (3) and (6).

5.2 Classification of Covariant and Invariant Instruments

One could ask whether the instruments of the form (15) exhaust all the possible instruments having the symmetry property (16). The answer is negative. Indeed, the following theorem can be proved.

Proposition 1 *Let $\mathcal{I} : \mathcal{B}(\mathbb{R}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ be an instrument satisfying the condition* (16). *Then there exist*:

- *a probability measure μ on $\mathbb{R}$*
- *a Hilbert space valued function $f \in L^2(\mathbb{R}^2, dx \otimes d\mu; \mathcal{H})$ satisfying $\|f\| = 1$*

such that

$$\begin{aligned} [\mathcal{I}_X(P_\phi)\psi](y) = \iiint & dy' dx d\mu(h) \chi_X(x) \phi(y+h) \overline{\phi(y'+h)} \\ & \times \langle f(y'-x+h, h) | f(y-x+h, h)\rangle \psi(y') \end{aligned} \tag{17}$$

for all ψ and $\phi \in \mathcal{H}$.

Conversely, for each choice of a probability measure μ on the real line and of a normalized function $f \in L^2(\mathbb{R}^2, dx \otimes d\mu; \mathcal{H})$, there is an instrument which satisfies the symmetry condition (16) *and whose action on pure states is given by* (17).

The observable associated to the instrument (17) *is the position observable Q_ρ determined by the probability measure*

$$d\rho(x) = \left(\int d\mu(h) dy \|\phi(y-x, h)\|^2\right) dx.$$

The above proposition is based on [14] and the proof will be given in a forthcoming paper [6].

As a consequence of Sect. 5.1 and Proposition 1, we conclude that for a position observable Q_ρ the following conditions are equivalent:

(i) The probability measure ρ is absolutely continuous with respect to the Lebesgue measure;
(ii) There is a Q_ρ-compatible instrument satisfying the symmetry property (16).

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